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SECOND ORDER RANDOM DIFFERENTIAL INCLUSIONS WITH BOUNDARY CONDITIONS

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ABSTRACT

The random differential inclusions is an important branch of probabilistic non-linear analysis and is applicable in classical as well as random phenomenon of the universe. In this paper, we have investigated the boundary value problem for second order random differential inclusions and proved an existence result through random fixed point theorem for condensing map of Martelli.

KEYWORDS:

Multi-valued map; Random differentialinclusions; condensing map; random fixedpoint theorem.

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1. INTRODUCTION

Consider the second order random boundary valued problem

$$x''(t,\omega) \in F((t,\omega), x(t,\omega)) \ , a.e.\ t \in J[0,T], \text{ for a. e } t \in J[0,T], \ \omega \in \Omega$$

$$L(x(0,\omega), x(T,\omega)) = 0 .$$

Where $F: J \times \mathrm{R} \times \Omega \to 2^{\mathrm{R}}$ a compact and convex valued random multivalued map and $L: J \times R^2 \times \Omega \to R$ is a continuous single valued map.

The upper and lower solutions has been successfully applied to study the existence of multiple solutions for boundary value problems of first order random differential inclusion. In the books of Bernfeld-Lakshmikantham [1], Ladde-Lakshmikantham-Vatsala [6], the thesis of De Coster [4], the papers of Carl-Heikkila-Kumpulainen [3], Cabada [2], Frigon-O'Regan [5], authors obtained in [2] and [3] existence results for differential inclusions with periodic boundary conditions for first and second order respectively. The existenc results for random differential inclusions are proved in paper of Palimkar[9,10].Biradar and Palimkar[11].In this paper, we investigate an existence result for the problem (1.1) – (1.2). Our methodology is based on a random fixed point theorem for condensing maps of Martelli [8].

2. AUXILIARY RESULTS

Consider $AC(J, \mathrm{R})$ is the space of all absolutely continuous functions $x: J \times \Omega \to R$

Condition $x \le \bar{x}$ if and only if $x(t,\omega) \le \bar{x}(t,\omega)$ for all $t \in J, \omega \in \Omega$.

Defines a partial ordering in $AC(J, \mathrm{R})$. If $\alpha, \beta \in AC(J, \mathrm{R})$ and $\alpha \le \beta$, we denote

$$[\alpha, \beta] = \{x \in AC(J, R) : \alpha \le x \le \beta\}$$

$W^{1.1}(J, \mathrm{R})$ denotes the Banach space of functions $x: J \times \Omega \to R$ which are absolutely continuous and whose derivative x' (which exists almost everywhere) is an element of $L^1(J, \mathrm{R})$ with the norm $\|y\| W^{1.1} = \|y\|_{L^1} + \|y'\|_{L^1}$ for all $y \in W^{1.1}(J, \mathrm{R})$.

Let $CC(x)$ denotes the set of all nonempty compact and convex subsets of X.

Definition 2.1. A random multivalued map $F: J \times \mathrm{R} \times \Omega \to 2^{\mathrm{R}}$ is said to be an L^1-random carathéodory if

(i) $(t,\omega) \to F((t,\omega), x)$ is random measurable for each $y \in \mathrm{R}, \omega \in \Omega$

(ii) $x \to F((t,\omega), x)$ is upper semi continuous for almost all $t \in J, \omega \in \Omega$

(iii) For each $k > 0$, there exists $h_k \in L^1(J, \Omega, R_+)$ such that

$\|F((t,\omega), x)\| = \sup\{|v| : v \in F((t,\omega), x)\} \le h_k(t,\omega)$ for all $|x| \le k$ and for almost all $t \in J, \omega \in \Omega$.

Definition 2.2. A function $x \in AC(J,R,\Omega)$ is said to be a solution of (1.1) – (1.2) if there existsa function $v \in L^1(J,R,\Omega)$ such that $v(t,\omega) \in F(t, x(t,\omega),\omega)$ a.e. on J, $x'(t,\omega) = v(t,\omega)$ a.e. on J and $L(x(0,\omega), x(t,\omega)) = 0$.

Definition 2.3. A function $\alpha \in AC(J,\mathrm{R})$ is said to be a lower solution of (1.1) – (1.2) if there exists $v_1 \in L^1(J,R,\Omega)$ such that $v_1(t,\omega) \in F((t,\omega),\alpha(t,\omega),\omega)$ a.e. on J, $J, \alpha'(t,\omega) \le v_1(t,\omega)$ a.e. on J and $L(\alpha(0,\omega),\alpha(T,\omega)) \le 0$

Similarly, a function $\beta \in AC(J,\mathrm{R})$ is said to be an upper solution of (1.1) – (1.2) if there exists $v_2 \in L^1(J,R,\Omega)$ such that $v_2(t,\omega) \in F((t,\omega),\beta(t,\omega),\omega)$ a.e. on $J, \beta'(t,\omega) \ge v_2(t,\omega)$ a.e. on J and $L(\beta(0,\omega),\beta(T,\omega)) \ge 0$

For the random multivalued map F and for each $x \in C(J,R,\Omega)$. we define $S^1_{F,x}$ by

$$S^1_{F,x} = \{v \in L^1(J,R,\Omega) : v(t,\omega) \in F((t,\omega), x(t,\omega)) \text{ for a.e. } t \in J \ , \omega \in \Omega \ \}$$

We quote the following lemmas useful for proving the the main result as

Lemma 2.1..Let I be a compact real interval and X be a Banach space. Let $F : I \times X \to CC(X); ((t,\omega), x) \to F((t,\omega), x)$ random measurable with respect to t for any $x \in X$ and u.s.c. with respect to y for almost each $t \in I$ and $S^1_{F,x} \ne \phi$ for any $y \in C(I,X)$ and let Γ be a linear continuous mapping from $L^1(I,X)$,to $C(I,X)$ then the operator

$$\Gamma o S^1_F : C(I,X) \to CC(C(I,X)), x \to (\Gamma o S^1_F)(x) = \Gamma(S^1_{F,x})$$

is a closed graph operator in $C(I,X) \times C(I,X)$.

Lemma 2.2. Let $G : X \to CC(X)$ be an u.s.c. condensing map. If the set

$$M := \{v \in X : \lambda v \in G(v) \ \text{ for some } \lambda > 1\}$$

is bounded, then G has a fixed point.

3. MAIN RESULT

Theorem 3.1.Suppose $F : J \times \mathrm{R} \times \Omega \to CC(\mathrm{R})$ is an L^1-Random carathéodory multi-valued map. Suppose the following conditions

(A_1) .There exist α and β in $W^{1.1}(J,R,\Omega)$ lower and upper solutions respectively for the

problem(1.1) – (1.2) such that $\alpha \le \beta$,

(A_2) Lisacontinuous single-valued map in $(x, y) \in [\alpha(0,\omega),\beta(0,\omega)] \times [\alpha(T,\omega),\beta(T,\omega)]$

and non increasing in $y \in [\alpha(T,\omega), \beta(T,\omega)]$, are satisfied. Then the problem (1.1) – (1.2) has at least one solution $y \in W^{1.1}(J,R,\Omega)$ such that

$$\alpha(t,\omega) \le x(t,\omega) \le \beta(t,\omega) \text{ for all } t \in J\ ,\omega \in \Omega .$$

Proof: Transform the problem into a random fixed point problem. Consider the following modified problem

$$x''(t,\omega) + y(t,\omega) \in F_1(t,\omega, y(t,\omega)\,, t \in J\ ,\omega \in \Omega \qquad (3.1)$$

$$x(0,\omega) = \tau((0,\omega), x(0,\omega) - L(\overline{x}(0,\omega), \overline{x}(T,\omega))) \qquad (3.2)$$

Where

$$F_1((t,\omega), x) = F((t,\omega), \tau((t,\omega), x)) + \tau((t,\omega), x),$$

$$\tau((t,\omega), x) = \max\{\alpha(t,\omega), \min\{x, \beta(t,\omega)\}\} \text{ , and}$$

$$\overline{x}(t,\omega) = \tau((t,\omega), x(t,\omega)).$$

Clearly a solution to (3.1) – (3.2) is a random fixed point of the operator $N : C(J,R,\Omega) \to 2^{C(J,R,\Omega)}$ defined by

$$N(x(\omega)) := \left\{ h \in C(J,R,\ \Omega) : h(t,\omega) = x(0,\omega) + \int_0^t [v(s,\omega) + \overline{x}(s,\omega) - x(s,\omega)]ds, v \in \tilde{S}^1_{F,\overline{x}}(\omega) \right\}$$

where

$$\tilde{S}^1_{F,\overline{x}}(\omega) = \left\{ v \in \tilde{S}^1_{F,\overline{x}}(\omega) : v(t,\omega) \ge v_1(t,\omega) \text{ a.e. on } H_1 \text{ and } v(t,\omega) \le v_2(t,\omega) \text{ a.e. on } H_2 \ \right\},$$

$$S^1_{F,\overline{x}}(\omega) = \left\{ v \in L^1(J,R,\Omega) : v(t,\omega) \in F((t,\omega), \overline{x}(t,\omega)) \text{ for a.e. } ,\omega \in \Omega\ , t \in J \right\},$$

$$H_1 = \left\{ t \in J : x(t,\omega) < \alpha(t,\omega) \le \beta(t,\omega) \right\},\ H_2 = \left\{ t \in J : \alpha(t,\omega) \le \beta(t,\omega) < x(t,\omega) \right\}.$$

We shall show that $N(\omega)$ is a completely continuous random multi-valued map, u.s.c. with convex closed values. The proof will be given in several steps.

Step 1: $N(x(\omega))$ is convex for each $x \in C(J,R,\Omega)$.

Indeed, if $h, \overline{h}$ belong to $N(x(\omega))$, then there exist $v \in \tilde{S}^1_{F,\overline{x}}(\omega)$ and $\overline{v} \in \tilde{S}^1_{F,\overline{x}}(\omega)$ such that

$$h(t,\omega) = x(0,\omega) + \int_0^t \left[v(s,\omega) + \overline{x}(s,\omega) - x(s,\omega) \right] ds,\ \ t \in J\ \ ,\omega \in \Omega$$

and

$$\overline{h}(t,\omega) = x(0,\omega) + \int_0^t \left[\overline{v}(s,\omega) + \overline{x}(s,\omega) - x(s,\omega) \right] ds,\ \ t \in J\ ,\omega \in \Omega$$

Let $0 \le k \le 1$. Then for each $t \in J\ ,\omega \in \Omega$ we have

$$[kh + (1-k)\overline{h}](t,\omega) = x(0,\omega) + \int_0^t [kv(s,\omega) + (1-k)\overline{v}(s,\omega) + \overline{x}(s) - x(s,\omega)]ds.$$

Since $S^{1}_{F,\bar{x}}(\omega)$ is convex then

$$kh+(1-k)\bar{h}\in G(x).$$

Step 2: $N(\omega)$ sends bounded sets into bounded sets in $C(J,R,\Omega)$.

Let $B_r=\{x\in C(J,R,\Omega):\|x\|_\infty\le r\}$, $(\|x\|_\infty=\sup\{|x(t,\omega)|:t\in J\}$ be a bounded set in $C(J,R,\Omega)$ and $x\in B_r$, then for each

$$|h(t,\omega)|\le|x(0,\omega)|+\int_0^t\left[|v(s,\omega)|+|\bar{x}(s,\omega)|-|x(s,\omega)|\right]ds,$$
$$B_r=\{x\in C(J,R,\Omega):\|x\|_\infty\le r\}$$
$$h\in N(x)(\omega)$$
$$\le\int_{u_1}^{u_2}|\phi_r(s,\omega)|\,ds+(u_2-u_1)\max(r,\sup_{t\in J}|\alpha(t,\omega)|,\sup_{t\in J}|\beta(t,\omega)|)+r(u_2-u_1).$$

there exists $v\in S^{1}_{F,\bar{x}}(\omega)$ such that

$$h(t,\omega)=x(0,\omega)+\int_0^t\left[v(s,\omega)+\bar{x}(s,\omega)-x(s,\omega)\right]ds,\ t\in J.,\omega\in\Omega$$

Thus for each $t\in J$, $\omega\in\Omega$, we get

$$|h(t,\omega)|\le|x(0,\omega)|+\int_0^t\left[|v(s,\omega)|+|\bar{x}(s,\omega)|-|x(s,\omega)|\right]ds,$$
$$\le\max(\alpha(0,\omega),\beta(0,\omega))+\|\phi_r\|_{L^1}+T\max(r,\sup_{t\in J}|\alpha(t,\omega)|,\sup_{t\in J}|\beta(t,\omega)|)+T_r$$

Step 3: N sends bounded sets in $C(J,R,\Omega)$ into equi-continuous sets.

Let $u_1,u_2\in J$, $u_1<u_2$, $B_r:=\{y\in C(J,R,\Omega):\|y\|_\infty\le r\}$ be a bounded set in $C(J,R,\Omega)$ and $y\in B_r$. For each $h\in N(x)(\omega)$ there exists $v\in S^{1}_{F,\bar{x}}(\omega)$ such that

$$h(t,\omega)=x(0,\omega)+\int_0^t[v(s,\omega)+\bar{x}(s,\omega)-x_n(s,\omega)]ds,\ t\in J.,\omega\in\Omega$$

We then have

$$|h(u_2)-h(u_1)|\le\int_{u_1}^{u_2}\left[|v(s,\omega)+\bar{x}(s,\omega)|+|x(s,\omega)|\right]ds$$

$$\le\int_{u_1}^{u_2}|\phi_r(s,\omega)|\,ds+(u_2-u_1)\max(r,\sup_{t\in J}|\alpha(t,\omega)|,\sup_{t\in J}|\beta(t,\omega)|)+r(u_2-u_1).$$

As a consequence of Step 2, Step 3 together with the Ascoli-Arzela theorem we can conclude that $N:C(J,R,\Omega)\to 2^{C(J,R,\Omega)}$ is a compact random multivalued map, and therefore, a condensing map.

Step 4: N has a closed graph.

Let $x_n \to x_0, h_n \in N(x_n)(\omega)$ and $h_n \to h_0$. We shall prove that $h_0 \in N(x_0)(\omega)$., $h_n \in N(x_n)(\omega)$ means that there exists $v_n \in S_{F,\bar{x}_n}(\omega)$ such that

$$h_n(t,\omega) = x(0,\omega) + \int_0^t [v_n(s,\omega) + \bar{x}_n(s,\omega) - x_n(s,\omega)]ds,\ t \in J\ ,\omega \in \Omega$$

We must prove that there exists $v_0 \in S^1_{F,\bar{x}_0}(\omega)$ such that

$$h_0(t,\omega) = x(0,\omega) + \int_0^t [v_0(s,\omega) + \bar{x}_0(s,\omega) - x_0(s,\omega)]ds,\ t \in J\ ,\omega \in \Omega$$

Consider the random linear continuous operator : $h_0(t,\omega) = x(0,\omega) + \int_0^t [v_0(s,\omega) + \bar{x}_0(s,\omega) - x_0(s,\omega)]ds,$ defined by

$$(\Gamma v)(t,\omega) = \int_0^t v(s,\omega)ds.$$

We have

$$\left\| \left(h_n - x(0,\omega) - \int_0^t [\bar{x}_n(s,\omega) - x_n(s,\omega)]ds \right) - \left(h_0 - x(0,\omega) - \int_0^t [\bar{x}_0(s,\omega) - x_0(s,\omega)]ds \right) \right\|_\infty \to 0$$

From Lemma 2.1, it follows that $\Gamma o S^1_F$ is a closed graph operator.

Also from the definition of Γ we have that

$$\left(h_n(t,\omega) - x(0,\omega) - \int_0^t [\overline{x_n}(s,\omega) - x_0(s,\omega)]ds \right) \in \Gamma\left(S^1_{F,\bar{x}_n}(\omega) \right).$$

Since $x_n \to x_0$ it follows from Lemma 2.1 that

$$h_0(t,\omega) = x(0,\omega) + \int_0^t [v_0(s,\omega) + \bar{x}_0(s,\omega) - x_0(s,\omega)]ds,\ t \in J \text{ for some } v_0 \in S^1_{F,\bar{x}_0}$$

Next we shall show that N (ω) has a random fixed point, by proving that

Step 5: The set

$$M = \{ v \in C(J,R,\Omega) : \lambda v \in N(v)(\omega) \text{ for some } \lambda > 1 \}$$

is bounded.

Let $x \in M$ then $\lambda x \in N(x)(\omega)$ for some $\lambda > 1$. Thus there exists $v \in S^1_{F,\bar{x}}(\omega)$ such that

$$x(t,\omega) = \lambda^{-1} x(0,\omega) + \lambda^{-1} \int_0^t \left[v(s,\omega) + \bar{x}(s,\omega) - x(s,\omega) \right] ds,\ t \in J\ ,\omega \in \Omega$$

Thus

$$|x(t,\omega)| \le |x(0,\omega)| + \int_0^t \left| v(s,\omega) + \bar{x}(s,\omega) - x(s,\omega) \right| ds,\ t \in J\ ,\omega \in \Omega.$$

From the definition of τ there exists $\phi \in L^1(J,R^+,\Omega)$ such that

$$\|F((t,\omega),\bar{x}(t,\omega))\| = \sup\{ |v| : v \in F((t,\omega),\bar{x}(t,\omega)) \} \le \phi(t,\omega) \text{ for each } x \in C(J,R,\Omega)\ ,\omega \in \Omega.$$

$$|x(t,\omega)| \leq \max(\alpha(0,\omega),\beta(0,\omega)) + \|\phi\|_{L^1} + T\max(\sup_{t\in J}|\alpha(t,\omega)|, \sup_{t\in J}|\beta(t,\omega)|) + \int_0^t |x(s,\omega)|ds.$$

Set $z_0 = \max(\alpha(0,\omega),\beta(0,\omega)) + \|\phi\|_{L^1} + T\max(\sup_{t\in J}|\alpha(t,\omega)|, \sup_{t\in J}|\beta(t,\omega)|)$

Using the Gronwall Lemma ,we get for each $t \in J$,$\omega \in \Omega$

$$|x(t,\omega)| \leq z_0 + z_0\int_0^t e^{t-s}ds$$

$$\leq z_0 + z_0(e^t - 1)$$

Thus,

$$\|x\|_\infty \leq z_0 + z_0(e^T - 1)$$

This shows that M is bounded.

Hence, Lemma 2.2 applies and N(ω) has a fixed point which is a solution to problem (3.1) –
(3.2).

Step 6: We shall show that the solution x of (3.1)-(3.2) satisfies

$$\alpha(t,\omega) \leq x(t,\omega) \leq \beta(t,\omega)\ t \in J\ ,\omega \in \Omega.$$

Let y be a solution to (3.1) – (3.2). We prove that

$$\alpha(t,\omega) \leq x(t,\omega)\ ,\ t \in J\ ,\omega \in \Omega$$

Suppose not. Then there exist $t_1, t_2 \in J, t_1 < t_2$ such that $\alpha(t_1,\omega) = y(t_1,\omega)$ and

$$\alpha(t,\omega) > x(t,\omega) \text{ for all } t \in (t_1,t_2).$$

In view of the definition of τ one has

$$x'(t,\omega) + x(t,\omega) \in F((t,\omega),\alpha(t,\omega)) + \alpha(t,\omega) \text{ a.e. on } (t_1,t_2).$$

Thus there exists $v(t,\omega) \in F((t,\omega),\alpha(t,\omega))$ a.e. on J with $v(t,\omega) \geq v_1(t,\omega)$ a.e. on (t_1,t_2) such that

$$x'(t,\omega) + x(t,\omega) = v(t,\omega) + \alpha(t,\omega) \text{ a.e. on } (t_1,t_2).$$

An integration on $(t_1,t]$, with $t \in (t_1,t_2)$ yields

$$x(t,\omega) - x(t_1,\omega) = \int_{t_1}^t [v(s,\omega) + (\alpha - x)(s,\omega)]ds$$

$$> \int_{t_1}^t v(s,\omega)ds.$$

Since α is a lower solution to (1.1) – (1.2), then

$$\alpha(t,\omega) - \alpha(t_1,\omega) \leq \int_{t_1}^t v_1(s,\omega)ds,\ t \in (t_1,t_2)\ ,\omega \in \Omega.$$

It follows from the facts $y(t_1,\omega)=\alpha(t_1,\omega), v(t,\omega)\geq v_1(t,\omega)$ that

$$\alpha(t,\omega)<x(t,\omega) \text{ for all } t\in(t_1,t_2)\ ,\omega\in\Omega .$$

which is a contradiction, since $x(t,\omega)<\alpha(t,\omega)$ for all $t\in(t_1,t_2)$. Consequently

$$\alpha(t,\omega)\leq x(t,\omega) \text{ for all } t\in J\ .,\omega\in\Omega$$

Clearly,we can prove that

$$x(t,\omega)\leq \beta(t,\omega) \text{ for all } t\in J\ .,\omega\in\Omega$$

This shows that the problem (3.1) – (3.2) has a solution in the interval $[\alpha,\beta]$.

Finally, we prove that every solution of (3.1) – (3.2) is also a solution to (1.1) – (1.2). We only need to show that

$$\alpha(0,\omega)\leq x(0,\omega)-L(\bar{x}(0,\omega),\bar{x}(T,\omega))\leq \beta(0,\omega) .$$

Notice first that we can prove that

$$\alpha(T,\omega)\leq x(T,\omega)\leq \beta(T,\omega) .$$

Suppose now that $x(0,\omega)-L(\bar{x}(0,\omega),\bar{x}(T,\omega))<\alpha(0,\omega).$ Then $x(0,\omega)=\alpha(0,\omega)$ and

$$x(0,\omega)-L(\alpha(0,\omega),\bar{x}(T,\omega))\leq \alpha(0,\omega) .$$

Since L is non increasing in y, we have

$$\alpha(0,\omega)\leq \alpha(0,\omega)-L(\alpha(0,\omega),\alpha(T,\omega))\leq \alpha(0,\omega)-L(\alpha(0,\omega),\bar{y}(T,\omega))<\alpha(0,\omega)$$

which is a contradiction.

Analogously we can prove that

$$x(0,\omega)-L(\tau(0,\omega),\tau(T,\omega))\leq \beta(0,\omega)$$

Then $x(t,\omega)$ is a random solution to (1.1) – (1.2).

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